

On a \mathbb{Z}_3 -Graded Generalization of the Witten Index

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Abstract

We construct a realization of the algebra of the \mathbb{Z}_3 -graded topological symmetry of type $(1, 1, 1)$ in terms of a pair of operators $D_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $D_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ satisfying $[D_1 D_1^\dagger, D_2^\dagger D_2] = 0$. We show that the sequence of the restriction of these operators to the zero-energy subspace forms a complex and establish the equality of the corresponding topological invariants with the analytic indices of these operators.

1 Introduction

Since the introduction of the supersymmetric quantum mechanics, there has been an on-going effort to develop and study its generalizations. Recently, we have offered in [1] a complete characterization of a class of generalizations of supersymmetry that share its intriguing topological properties. These symmetries that are called topological symmetries involve certain integer-valued topological invariants that are the analogs of the Witten index of supersymmetry. In Ref. [1], we have presented a crude description of the topological invariants associated with a class of topological symmetries in terms of certain operators, $D_\ell : \mathcal{H}_\ell \rightarrow \mathcal{H}_{\ell+1}$. These operators are however subject to a number of rather complicated compatibility conditions. The purpose of this article is to give a nontrivial solution of these compatibility conditions for the case of \mathbb{Z}_3 -graded topological symmetry

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of type $(1, 1, 1)$. In view of the fact that supersymmetry coincides with the \mathbb{Z}_2 -graded topological symmetry of type $(1, 1)$, \mathbb{Z}_3 -graded topological symmetry of type $(1, 1, 1)$ is a simplest generalization of supersymmetry.

The main motivation for the study of these symmetries is to find out whether they lead to more general topological invariants than the Witten index. The latter is known to be identical with the analytic index of a Fredholm operator, [2]. This has been the key observation in developing the supersymmetric proofs of the Atiyah-Singer index theorem [3]. Historically, the fact that the analytic index of a Fredholm operator is a topological invariant was known much earlier than the actual proof of the index theorem. See for example [4]. Therefore, a relevant question is whether the \mathbb{Z}_3 -graded topological symmetry of type $(1, 1, 1)$ leads to a topological invariant that is more general than the analytic index of a Fredholm operator.

The organization of this article is as follows. In Section 2, we recall the definition of the topological symmetries and review their basic properties. In section 3, we consider the \mathbb{Z}_3 -graded topological symmetry of type $(1, 1, 1)$ in its three-component representation and derive the compatibility conditions for the operators appearing in this representation. In Section 4, we discuss a particular nontrivial solution of these compatibility conditions and describe the corresponding topological invariants. In Section 5, we present our concluding remarks.

2 Topological Symmetries

We begin our analysis by a review of the basic properties of topological symmetries. A more detailed discussion can be found in [1].

A quantum system is said to possess a \mathbb{Z}_n -graded topological symmetry (TS) of type (m_1, m_2, \dots, m_n) iff the following conditions are satisfied.

1. The quantum system is \mathbb{Z}_n -graded. This means that the Hilbert space \mathcal{H} of the quantum system is the direct sum of n of its (nontrivial) subspaces \mathcal{H}_ℓ , and its

Hamiltonian has a complete set of eigenvectors with definite *color* or *grading*. (A state is said to have a definite grading c_ℓ iff it belongs to \mathcal{H}_ℓ);

2. The energy spectrum is nonnegative;
3. For every eigenvalue E there corresponds a positive integer λ_E such that E is $\sum_{\ell=1}^n m_\ell \lambda_E$ -fold degenerate, and the corresponding eigenspaces are spanned by $m_1 \lambda_E$ vectors of grade c_1 , $m_2 \lambda_E$ vectors of grade c_2 , \dots , and $m_n \lambda_E$ vectors of grade c_n .

For a system with a TS we can introduce a set of integer-valued topological invariants, namely

$$\Delta_{ij} := m_i n_j^{(0)} - m_j n_i^{(0)}, \quad (1)$$

where $i, j \in \{1, 2, \dots, n\}$ and $n_\ell^{(0)}$ denotes the number of zero-energy states of grade c_ℓ .

It turns out that using the above definition of TS, one can obtain the underlying operator algebras supporting these symmetries. In particular, \mathbb{Z}_2 -graded TS of type $(1, 1)$ coincides with supersymmetry and Δ_{11} yields the Witten index. Similarly, one obtains the algebras of parasupersymmetry of order 2 and fractional supersymmetry of arbitrary order as special cases of TSs.

In this article we shall confine our attention to the case of \mathbb{Z}_3 -graded TS of type $(1, 1, 1)$. Following [1], we grade the state vectors by associating them with a third root of unity, i.e., we use a grading operator satisfying

$$\tau^3 = 1, \quad \tau^\dagger = \tau^{-1}, \quad [H, \tau] = 0, \quad [\tau, \mathcal{Q}]_q = 0, \quad (2)$$

where \mathcal{Q} is the generator of TS, $q := e^{2\pi i/3}$, and $[\ , \]_q$ stands for the q -commutator, $[O_1, O_2]_q := O_1 O_2 - q O_2 O_1$. The operator algebra for \mathbb{Z}_3 -graded TS of type $(1, 1, 1)$ has the form

$$\mathcal{Q}^3 = K, \quad (3)$$

$$Q_1^3 + M Q_1 = 2^{-3/2} (K + K^\dagger), \quad (4)$$

$$Q_2^3 + M = 2^{-3/2} i^3 (-K + K^\dagger), \quad (5)$$

where K and M are operators commuting with all other operators, M is self-adjoint, and

$$Q_1 := \frac{\mathcal{Q} + \mathcal{Q}^\dagger}{\sqrt{2}}, \quad Q_2 := \frac{\mathcal{Q} - \mathcal{Q}^\dagger}{\sqrt{2}i}. \quad (6)$$

For $K = H$, Eq. (3) coincides with the defining relation for the fractional supersymmetry of order 3, [5].

It is not difficult to show [1] that Eq. (3) implies the particular degeneracy structure of the \mathbb{Z}_3 -graded TS of type $(1, 1, 1)$ provided that $\ker(K) \subseteq \ker(H)$. In particular, a quantum system with a Hamiltonian H has a \mathbb{Z}_3 -graded TS of type $(1, 1, 1)$ if the following conditions are satisfied

1. There are operators τ and \mathcal{Q} satisfying Eqs. (2);
2. The spectrum of H is nonnegative;
3. Eq. (3) holds for $K = H$.

The presence of this particular topological symmetry in turn implies the existence of a self-adjoint operator M that commutes with τ and \mathcal{Q} and fulfills Eqs. (4) and (5).

Hereafter, we shall set $K = H$.

3 Three-Component Realization of \mathbb{Z}_3 -graded TS of type $(1, 1, 1)$

Similarly to the case of supersymmetric quantum mechanics, we can obtain a realization of the \mathbb{Z}_3 -graded TS by identifying the state vectors with column vectors whose rows have definite grade. We first let $\mathcal{H}_1, \mathcal{H}_2$, and \mathcal{H}_3 be three Hilbert spaces and define the total Hilbert space \mathcal{H} as their inner sum, $\mathcal{H} := \oplus_{\ell=1}^3 \mathcal{H}_\ell$. Every state vector $|\psi\rangle \in \mathcal{H}$ can be written as the sum of its components $|\psi\rangle_\ell$ belonging to \mathcal{H}_ℓ . In the three-component representation of \mathcal{H} , this is expressed as

$$|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \\ |\psi_3\rangle \end{pmatrix}.$$

In this representation, τ is defined by

$$\tau := \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (7)$$

and the operators \mathcal{Q}, H, M take the form

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & D_3 \\ D_1 & 0 & 0 \\ 0 & D_2 & 0 \end{pmatrix}, \quad (8)$$

$$H = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}, \quad (9)$$

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \quad (10)$$

where $D_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $D_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_3$, $D_3 : \mathcal{H}_3 \rightarrow \mathcal{H}_1$, $H_\ell : \mathcal{H}_\ell \rightarrow \mathcal{H}_\ell$, and $M_\ell : \mathcal{H}_\ell \rightarrow \mathcal{H}_\ell$, with $\ell \in \{1, 2, 3\}$, are linear operators, and H_ℓ and M_ℓ are self-adjoint. Eqs. (8), (9), and (10) are direct consequences of Eqs. (2).

The condition that H and M commute with \mathcal{Q} is equivalent to the statement that (H_1, H_2, H_3) and (M_1, M_2, M_3) are triplets of isospectral operators, as substituting (9), (8), and (10) in $[H, \mathcal{Q}] = 0$ and $[M, \mathcal{Q}] = 0$ yields

$$D_1 H_1 = H_2 D_1, \quad D_2 H_2 = H_3 D_2, \quad D_3 H_3 = H_1 D_3, \quad (11)$$

$$D_1 M_1 = M_2 D_1, \quad (12)$$

$$D_2 M_2 = M_3 D_2, \quad (13)$$

$$D_3 M_3 = M_1 D_3, \quad (14)$$

Note that, in view of Eqs. (9) and (11), in order to ensure that the spectrum of H is nonnegative it is sufficient to show that one of H_ℓ has a nonnegative spectrum.

Next, we substitute (8) and (9) in Eq. (3) with $K = H$, i.e., set $\mathcal{Q}^3 = H$. This yields

$$H_1 = D_3 D_2 D_1, \quad H_2 = D_1 D_3 D_2, \quad H_3 = D_2 D_1 D_3. \quad (15)$$

Clearly, Eqs. (15) agree with Eqs. (11). However, the condition that H is self-adjoint implies $H_\ell = H_\ell^\dagger$. This relation restricts the choice of the operators D_ℓ and leads to the first set of compatibility conditions for D_ℓ , namely

$$D_1^\dagger D_2^\dagger D_3^\dagger = D_3 D_2 D_1, \quad (16)$$

$$D_2^\dagger D_3^\dagger D_1^\dagger = D_1 D_3 D_2, \quad (17)$$

$$D_3^\dagger D_1^\dagger D_2^\dagger = D_2 D_1 D_3. \quad (18)$$

Next, we use Eqs. (8) and (6) to express Q_1 and Q_2 in the three-component representation. Substituting the result for Q_1 in Eq. (4) and making use of (10), we find

$$M_1 D_3 = D_3 D_2 D_2^\dagger + (D_3 D_3^\dagger + D_1^\dagger D_1) D_3, \quad (19)$$

$$M_3 D_2 = D_2 D_1 D_1^\dagger + (D_2 D_2^\dagger + D_3^\dagger D_3) D_2, \quad (20)$$

$$M_2 D_1 = D_2 D_2 D_3^\dagger + (D_1 D_1^\dagger + D_2^\dagger D_2) D_1. \quad (21)$$

Combining these equations with Eqs. (12) – (14), we find the second set of compatibility conditions for D_ℓ :

$$D_2^\dagger D_2 D_1 D_3 = D_1 D_3 D_2 D_2^\dagger, \quad (22)$$

$$D_3^\dagger D_3 D_2 D_1 = D_2 D_1 D_3 D_3^\dagger, \quad (23)$$

$$D_1^\dagger D_1 D_3 D_2 = D_3 D_2 D_1 D_1^\dagger. \quad (24)$$

Finally, we substitute the matrix representation of Q_2 and Eq. (10) in Eq. (5). The resulting matrix equation is trivially satisfied provided that we enforce (22) – (24). Therefore, Eq. (5) does not lead to further conditions on D_ℓ .

We wish to emphasize that so far we obtained a set of necessary conditions, that is (16) – (18) and (22) – (24), for having a \mathbb{Z}_3 -graded TS of type $(1, 1, 1)$. These conditions are not sufficient as we must also make sure that the energy spectrum is nonnegative. Furthermore, as we mentioned above, Eq. (3) with the choice of $K = H$ is sufficient for having the necessary degeneracy structure for the \mathbb{Z}_3 -graded TS of type $(1, 1, 1)$. The

existence of M and consequently M_ℓ follows from the definition of the TS. Hence the conditions (22) – (23) should be automatically satisfied provided that we can fulfill (16) – (18). This in turns means that if we can obtain a solution of (16) – (18) and show that the energy spectrum is nonnegative, then

$$\Delta_{ij} = n_j^{(0)} - n_i^{(0)} \quad (25)$$

are topological invariants.

In view of Eqs. (9) and (15), we also have $n_\ell^{(0)} = \dim(\ker H_\ell)$ and

$$\Delta_{12} = -\Delta_{21} = \dim(\ker D_1 D_3 D_2) - \dim(\ker D_3 D_2 D_1), \quad (26)$$

$$\Delta_{23} = -\Delta_{32} = \dim(\ker D_2 D_1 D_3) - \dim(\ker D_1 D_3 D_2), \quad (27)$$

$$\Delta_{13} = -\Delta_{31} = \Delta_{12} + \Delta_{23}, \quad (28)$$

where ‘dim’ and ‘ker’ abbreviate ‘dimension’ and ‘kernel’, respectively.

4 Solution of the Compatibility Conditions

In Ref. [1], we gave a simple solution of the compatibility conditions, namely

$$D_2 = D_1^\dagger, \quad D_3 = 1, \quad (29)$$

which applies to the case that $\mathcal{H}_3 = \mathcal{H}_1$ and D_1 is a Fredholm operator. One can easily check that all the compatibility conditions are satisfied for this choice of D_ℓ . The corresponding independent invariants are given by

$$\Delta_{12} = -\Delta_{23} = - \left[\dim(\ker D_1) - \dim(\ker D_1^\dagger) \right] = - \text{Analytic index } (D_1),$$

$$\Delta_{13} = 0,$$

where we have made use of Eqs. (26) – (28, (29) and the fact that for any Fredholm operator D ,

$$\ker(D^\dagger D) = \ker(D). \quad (30)$$

In the following we construct a more interesting solution of the compatibility conditions for D_ℓ which applies for arbitrary choices of \mathcal{H}_ℓ . This solution corresponds to the choice

$$D_3 = (D_2 D_1)^\dagger = D_1^\dagger D_2^\dagger. \quad (31)$$

One can check that for this choice of D_3 conditions (16) and (18) are automatically fulfilled, whereas (17) imposes the following condition on D_1 and D_2 .

$$[D_1 D_1^\dagger, D_2^\dagger D_2] = 0. \quad (32)$$

It turns out that this is the only condition on the operators D_1 and D_2 , as Eqs. (22) – (23) are trivially satisfied provided that (32) holds. This confirms our earlier remark that (22) – (23) should follow if we can satisfy (16) – (18).

Furthermore, using (12) and (13) and setting

$$M_2 = D_1 D_1^\dagger + D_2^\dagger D_2 + D_1 D_1^\dagger D_2^\dagger D_2, \quad (33)$$

we can also satisfy (19) – (21).

Equation (31) also implies that the Hamiltonian H has a nonnegative spectrum. In order to see this we substitute Eq. (31) in Eqs. (15). This yields

$$H_1 = D_3^\dagger D_3, \quad H_2 = D_1 D_1^\dagger D_2^\dagger D_2, \quad H_3 = D_3 D_3^\dagger. \quad (34)$$

Clearly these isospectral operators have a nonnegative spectrum, and the energy spectrum is nonnegative.

Next we compute the topological invariants associated with this solution. In view of Eqs. (30), (32) and (26) – (28), and the identity

$$\ker(D_1 D_1^\dagger D_2^\dagger D_2) = \ker(D_1^\dagger D_2^\dagger D_2),$$

we have

$$\Delta_{12} = \dim \left[\ker(D_1^\dagger D_2^\dagger D_2) \right] - \dim \left[\ker D_2 D_1 \right], \quad (35)$$

$$\Delta_{23} = \dim \left[\ker (D_2 D_1)^\dagger \right] - \dim \left[\ker(D_1^\dagger D_2^\dagger D_2) \right], \quad (36)$$

$$\Delta_{13} = - \text{Analytic Index } (D_2 D_1). \quad (37)$$

In summary, we have so far shown that for any two operators D_1 and D_2 satisfying condition (32), the quantities (35) – (37) are topological invariants.

Next, we study the properties of the restrictions of the operators D_i to the zero-energy subspaces

$$\mathcal{H}_i^{(0)} := \ker(H_i) \quad (38)$$

of \mathcal{H}_i . First we note that according to Eqs. (15) – (18) and (38),

$$\ker(D_1) \subseteq \ker(H_1) =: \mathcal{H}_1^{(0)}, \quad \ker(D_2) \subseteq \ker(H_2) =: \mathcal{H}_2^{(0)}, \quad (39)$$

$$\ker(D_1^\dagger) \subseteq \ker(H_2) =: \mathcal{H}_2^{(0)}, \quad \ker(D_2^\dagger) \subseteq \ker(H_3) =: \mathcal{H}_3^{(0)}. \quad (40)$$

Furthermore in view of (29), (34), and (38), we can easily show that for every $|\psi_i\rangle \in \mathcal{H}_i^{(0)}$,

$$H_1|\psi_1\rangle = D_3 D_3^\dagger |\psi_1\rangle = 0 \Leftrightarrow D_2 D_1 |\psi_1\rangle = D_3^\dagger |\psi_1\rangle = 0, \quad (41)$$

$$H_2|\psi_2\rangle = D_1 D_1^\dagger D_2^\dagger D_2 |\psi_2\rangle = 0 \Leftrightarrow D_1^\dagger D_2^\dagger D_2 |\psi_2\rangle = 0, \quad (42)$$

$$H_3|\psi_3\rangle = D_3^\dagger D_3 |\psi_3\rangle = 0 \Leftrightarrow (D_2 D_1)^\dagger |\psi_3\rangle = D_3 |\psi_3\rangle = 0. \quad (43)$$

According to these identities and relations (39), $D_1|\psi_1\rangle \in \ker(D_2)$, $D_2|\psi_2\rangle \in \ker(D_3)$.

Therefore, if we denote the restriction of D_i to $\mathcal{H}_i^{(0)}$ by $D_i^{(0)}$, we have

$$\text{im}(D_1^{(0)}) \subseteq \ker(D_2^{(0)}), \quad \text{im}(D_2^{(0)}) \subseteq \ker(D_3^{(0)}), \quad \text{im}(D_3^{(0)}) = \{0\}, \quad (44)$$

where ‘im’ abbreviates ‘image.’ In other words, the sequence

$$\{0\} \hookrightarrow \mathcal{H}_1^{(0)} \xrightarrow{D_1^{(0)}} \mathcal{H}_2^{(0)} \xrightarrow{D_2^{(0)}} \mathcal{H}_3^{(0)} \xrightarrow{D_3^{(0)}} \{0\} \quad (45)$$

is indeed a complex, and we can define the corresponding cohomology groups $\ker(D_{i+1}^{(0)})/\text{im}(D_i^{(0)})$ and Betti numbers

$$b_i := \dim[\ker(D_{i+1}^{(0)})/\text{im}(D_i^{(0)})] = \dim[\ker(D_{i+1}^{(0)})] - \dim[\text{im}(D_i^{(0)})] \quad (46)$$

Next, we note that $\mathcal{H}_i^{(0)}$ are assumed to be finite-dimensional. This implies that for $i \in \{1, 2\}$

$$\dim[\ker(D_i^{(0)})] + \dim[\text{im}(D_i^{(0)})] = \dim[\mathcal{H}_i^{(0)}], \quad (47)$$

$$\dim[\text{coker}(D_i^{(0)})] + \dim[\text{im}(D_i^{(0)})] = \dim[\mathcal{H}_{i+1}^{(0)}], \quad (48)$$

where ‘coker’ stands for ‘cokernel.’ In view of Eqs. (25), (47) and (48), we then obtain

$$\Delta_{i,i+1} = n_{i+1}^{(0)} - n_i^{(0)} = \dim[\text{coker}(D_i^{(0)})] - \dim[\ker(D_i^{(0)})] = \dim[\ker(D_i^{(0)\dagger})] - \dim[\ker(D_i^{(0)})]. \quad (49)$$

Note however that because $\ker(D_i) \subseteq \mathcal{H}_i^{(0)}$ and $\ker(D_i^\dagger) \subseteq \mathcal{H}_{i+1}^{(0)}$, we have

$$\ker(D_i^{(0)}) = \ker(D_i), \quad \ker(D_i^{(0)\dagger}) = \ker(D_i^\dagger). \quad (50)$$

Combining Eqs. (49) and (50), we finally obtain, for $i \in \{1, 2\}$,

$$\Delta_{i,i+1} = - \text{Analytic Index } (D_i). \quad (51)$$

This is our main result. It indicates that for the solution (29) the \mathbb{Z}_3 -graded generalization of the Witten index is nothing but the analytic index of a Fredholm operator. A corollary of this result is the following.

Corollary: Let $D_i : \mathcal{H}_i \rightarrow \mathcal{H}_{i+1}$, for $i \in \{1, 2\}$, be Fredholm operators satisfying $[D_1 D_1^\dagger, D_2^\dagger D_2] = 0$. Then

$$\dim[\ker D_2 D_1] - \dim[\ker(D_1^\dagger D_2^\dagger D_2)] = \text{Analytic Index } (D_1), \quad (52)$$

$$\dim[\ker(D_1^\dagger D_2^\dagger D_2)] - \dim[\ker(D_2 D_1)^\dagger] = \text{Analytic Index } (D_2), \quad (53)$$

$$\text{Analytic Index } (D_1) + \text{Analytic Index } (D_2) = \text{Analytic Index } (D_2 D_1). \quad (54)$$

Proof: Eqs. (52) and (53) are direct consequences of Eqs. (35), (36), (28), and (51).

Eq. (54) follows from Eqs. (28), (52) and (53). \square

5 Conclusion

In this article we have explored the topological invariants of a certain class of \mathbb{Z}_3 -graded topological symmetries of type $(1, 1, 1)$. The operator algebra of \mathbb{Z}_3 -graded topological symmetries of type $(1, 1, 1)$ together with the choice $K = H$ and the condition of the non-negativity of the energy spectrum imply the presence of this topological symmetry and

ensures the topological invariance of Δ_{ij} . This makes the construction of concrete realizations of this symmetry more tractable. Using the algebraic structure and manipulating the resulting compatibility conditions, we found a nontrivial solution of these conditions. We explored the meaning of the associated topological invariants and showed that they are related to the analytic indices of Fredholm operators. Therefore, for the realization of the \mathbb{Z}_3 -graded topological symmetries of type $(1, 1, 1)$ that we considered, the generalized Witten index does not yield a more general topological invariant. This situation is very similar to the topological invariants of $p = 2$ parasupersymmetry (alternatively \mathbb{Z}_2 -graded topological symmetry of type $(1, 2)$) studied in Ref. [6]. There also the known realization of the $p = 2$ parasupersymmetry does not lead to a more general invariant.

Our investigation of the nature of the topological invariants associated with \mathbb{Z}_3 -graded topological symmetries of type $(1, 1, 1)$ – which is a graded fractional supersymmetry of order 3 — has also revealed the following interesting properties.

1. The graded Hamiltonians H_ℓ are isospectral. This is very similar to the case of supersymmetry and may be useful in obtaining exactly solvable potentials.
2. The restriction of the square of the symmetry generator \mathcal{Q} to the zero-energy eigensubspace \mathcal{H}_0 vanishes identically. This in turn implies $\ker \mathcal{Q}^2 = \mathcal{H}_0$.

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